Vertiefungskurs Mathematik Klasse 12

Lösungen: Aufgaben zu Linienintegralen

AUFGABE 1

a) $f(x) = \sqrt{4-x^2}$ ➔ $f'(x) = \frac{-2x}{2\cdot\sqrt{4-x^2}} = -\frac{x}{\sqrt{4-x^2}}$

Länge L des Kurvenstückes: $L = \int_0^2 \sqrt{1+(f'(x))^2}dx = \int_0^2 \sqrt{1+\frac{x^2}{4-x^2}}dx$

$$L = \int_0^2 \sqrt{\frac{4}{4-x^2}}dx$$

Substitution: $x = 2\sin(t)$ ➔ $\frac{dx}{dt} = 2\cos(t)$ ➔ $dx = 2\cos(t)\,dt$

Grenzen: $t_1 = 0$ und $t_2 = \frac{\pi}{2}$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\frac{4}{4-4(\sin(t))^2}}\cdot 2\cos(t)\,dt = \int_0^{\frac{\pi}{2}} \sqrt{\frac{4}{4(\cos(t))^2}}\cdot 2\cos(t)\,dt = \int_0^{\frac{\pi}{2}} \frac{2\cos(t)}{\cos(t)}dt$$

$$L = \int_0^{\frac{\pi}{2}} 2dt = [2t]_0^{\frac{\pi}{2}} = \pi$$

b) $g(x) = \cosh(x)$ ➔ $g'(x) = \sinh(x)$

Länge L des Kurvenstückes: $L = \int_0^2 \sqrt{1+(g'(x))^2}dx = \int_0^2 \sqrt{1+(\sinh(x))^2}dx$

$$L = \int_0^2 \cosh(x)\,dx = [\sinh(x)]_0^2 = \sinh(2) - \sinh(0) = \sinh(2) \approx 3{,}63$$

c) $h(x) = 2\sqrt{x}$ ➔ $h'(x) = \frac{2}{2\cdot\sqrt{x}} = \frac{1}{\sqrt{x}}$

Länge L des Kurvenstückes: $L = \int_1^4 \sqrt{1+(h'(x))^2}dx = \int_1^4 \sqrt{1+\frac{1}{x}}dx$

Substitution: $u^2 = 1+\frac{1}{x}$ ➔ $x = \frac{1}{u^2-1}$

Partialbruchzerlegung: $x = \frac{1}{u^2-1} = \frac{1}{(u-1)\cdot(u+1)} = \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1} = \frac{1}{2}\cdot\left(\frac{1}{u-1} - \frac{1}{u+1}\right)$

$\frac{dx}{du} = \frac{1}{2}\cdot\left(-\frac{1}{(u-1)^2} + \frac{1}{(u+1)^2}\right)$ ➔ $dx = \frac{1}{2}\cdot\left(-\frac{1}{(u-1)^2} + \frac{1}{(u+1)^2}\right)du$

Grenzen: $u_1 = \sqrt{2}$ und $u_2 = \sqrt{\frac{5}{4}}$

$$L=\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\sqrt{u^2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{(u-1)^2}+\frac{1}{(u+1)^2}\right)du=\frac{1}{2}\cdot\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\left(-\frac{u}{(u-1)^2}+\frac{u}{(u+1)^2}\right)du$$

$$L=\frac{1}{2}\cdot\left(\underbrace{\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{u}{(u+1)^2}du}_{I_1}-\underbrace{\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{u}{(u-1)^2}du}_{I_2}\right)$$

$$I_1=\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{u}{(u+1)^2}du=\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{u+1-1}{(u+1)^2}du=\underbrace{\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{1}{u+1}du}_{I_{1;a}}-\underbrace{\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{1}{(u+1)^2}du}_{I_{1;b}}$$

$$I_{1;a}=\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{1}{u+1}du=[ln(u+1)]_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}=ln\left(\sqrt{\frac{5}{4}}+1\right)-ln(\sqrt{2}+1)$$

$$I_{1;b}=\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{1}{(u+1)^2}du=\left[-\frac{1}{u+1}\right]_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}=-\frac{1}{\sqrt{\frac{5}{4}}+1}+\frac{1}{\sqrt{2}+1}$$

$$I_1=ln\left(\sqrt{\frac{5}{4}}+1\right)-ln(\sqrt{2}+1)+\frac{1}{\sqrt{\frac{5}{4}}+1}-\frac{1}{\sqrt{2}+1}$$

$$I_2=\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{u}{(u-1)^2}du=\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{u-1+1}{(u-1)^2}du=\underbrace{\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{1}{u-1}du}_{I_{2;a}}+\underbrace{\int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}}\frac{1}{(u-1)^2}du}_{I_{2;b}}$$

$$I_{2;a} = \int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}} \frac{1}{u-1}\,du = [ln(u-1)]_{\sqrt{2}}^{\sqrt{\frac{5}{4}}} = ln\left(\sqrt{\frac{5}{4}}-1\right) - ln\left(\sqrt{2}-1\right)$$

$$I_{2;b} = \int_{\sqrt{2}}^{\sqrt{\frac{5}{4}}} \frac{1}{(u-1)^2}\,du = \left[-\frac{1}{u-1}\right]_{\sqrt{2}}^{\sqrt{\frac{5}{4}}} = -\frac{1}{\sqrt{\frac{5}{4}}-1} + \frac{1}{\sqrt{2}-1}$$

$$I_2 = ln\left(\sqrt{\frac{5}{4}}-1\right) - ln\left(\sqrt{2}-1\right) - \frac{1}{\sqrt{\frac{5}{4}}-1} + \frac{1}{\sqrt{2}-1}$$

$$I_1 - I_2 = ln\left(\sqrt{\frac{5}{4}}+1\right) - ln\left(\sqrt{\frac{5}{4}}-1\right) - ln\left(\sqrt{2}+1\right) + ln\left(\sqrt{2}-1\right) + \frac{1}{\sqrt{\frac{5}{4}}+1}$$

$$+\frac{1}{\sqrt{\frac{5}{4}}-1} - \frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{2}-1}$$

Anwenden von Logarithmengesetzen und Rationalmachen der Nenner liefert:

$$ln\left(\sqrt{\frac{5}{4}}+1\right) - ln\left(\sqrt{\frac{5}{4}}-1\right) = ln\left(\frac{\sqrt{\frac{5}{4}}+1}{\sqrt{\frac{5}{4}}-1}\right) = ln\left(\frac{\left(\sqrt{\frac{5}{4}}+1\right)^2}{\frac{1}{4}}\right)$$

$$= ln\left(4\cdot\left(\frac{1}{2}\cdot\sqrt{5}+1\right)^2\right) = ln\left(\left(2\cdot\left(\frac{1}{2}\cdot\sqrt{5}+1\right)\right)^2\right) = 2\cdot ln\left(\sqrt{5}+2\right)$$

$$ln\left(\sqrt{2}-1\right) - ln\left(\sqrt{2}+1\right) = ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = ln\left(\frac{\left(\sqrt{2}-1\right)^2}{1}\right) = 2\cdot ln\left(\sqrt{2}-1\right)$$

$$\frac{1}{\sqrt{\frac{5}{4}}+1} + \frac{1}{\sqrt{\frac{5}{4}}-1} = \frac{\sqrt{\frac{5}{4}}-1+\sqrt{\frac{5}{4}}+1}{\left(\sqrt{\frac{5}{4}}+1\right)\cdot\left(\sqrt{\frac{5}{4}}-1\right)} = \frac{2\cdot\frac{1}{2}\cdot\sqrt{5}}{\frac{1}{4}} = 4\cdot\sqrt{5}$$

$$-\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{2}-1} = -\left(\frac{\sqrt{2}-1+\sqrt{2}+1}{\left(\sqrt{2}+1\right)\cdot\left(\sqrt{2}-1\right)}\right) = -\frac{2\cdot\sqrt{2}}{1} = -2\cdot\sqrt{2}$$

$I_1 - I_2 = 2 \cdot ln(\sqrt{5}+2) + 2 \cdot ln(\sqrt{2}-1) + 4 \cdot \sqrt{5} - 2 \cdot \sqrt{2}$

$I_1 - I_2 = 2 \cdot \left(ln(\sqrt{5}+2) + ln(\sqrt{2}-1) + 2 \cdot \sqrt{5} - \sqrt{2}\right)$

$L = \frac{1}{2} \cdot (I_1 - I_2) = ln\left(\sqrt{5}+2\right) + ln\left(\sqrt{2}-1\right) + 2 \cdot \sqrt{5} - \sqrt{2} \approx 3{,}62$

AUFGABE 2

a) $f(x) = \sqrt{x^3} = x^{1,5}$ ➔ $f'(x) = \frac{3}{2} x^{0,5} = \frac{3}{2}\sqrt{x}$

Länge L des Kurvenstückes: $L = \int_1^4 \sqrt{1 + (f'(x))^2}dx = \int_1^4 \sqrt{1 + \frac{9}{4}x}dx$

Substitution: $u = 1 + \frac{9}{4}x$ ➔ $x = \frac{4}{9} \cdot (u-1)$

$\frac{dx}{du} = \frac{4}{9}$ ➔ $dx = \frac{4}{9}du$

Grenzen: $u_1 = \frac{13}{4}$ und $u_2 = 10$

$$L = \int_{\frac{13}{4}}^{10} \sqrt{u} \cdot \frac{4}{9}du = \frac{4}{9} \cdot \int_{\frac{13}{4}}^{10} u^{0,5}du = \frac{4}{9} \cdot \left[\frac{2}{3} \cdot u^{1,5}\right]_{\frac{13}{4}}^{10} = \frac{8}{27} \cdot \left[\sqrt{u^3}\right]_{\frac{13}{4}}^{10}$$

$$L = \frac{8}{27} \cdot \left(10 \cdot \sqrt{10} - \frac{13}{4} \cdot \sqrt{\frac{13}{4}}\right) = \frac{8}{27} \cdot \left(10 \cdot \sqrt{10} - \frac{13}{8} \cdot \sqrt{13}\right) \approx 7{,}63$$

b) $L = \int_4^a \sqrt{1 + (f'(x))^2}dx = \int_4^a \sqrt{1 + \frac{9}{4}x}dx = 50$

Aus a) folgt:

$$L = \int_{10}^{1+\frac{9}{4}a} \sqrt{u} \cdot \frac{4}{9}du = \frac{8}{27} \cdot \left[\sqrt{u^3}\right]_{10}^{1+\frac{9}{4}a} = \frac{8}{27} \cdot \left(\left(1 + \frac{9}{4}a\right)^{\frac{3}{2}} - 10 \cdot \sqrt{10}\right) = 50$$

➔ $\left(1 + \frac{9}{4}a\right)^{\frac{3}{2}} - 10 \cdot \sqrt{10} = \frac{675}{4}$ ➔ $\left(1 + \frac{9}{4}a\right)^{\frac{3}{2}} = \frac{675}{4} + 10 \cdot \sqrt{10}$

$1 + \frac{9}{4}a = \left(\frac{675}{4} + 10 \cdot \sqrt{10}\right)^{\frac{2}{3}}$ ➔ $a = \frac{4}{9} \cdot \left(\left(\frac{675}{4} + 10 \cdot \sqrt{10}\right)^{\frac{2}{3}} - 1\right) \approx 14{,}77$

➔ $f(a) = \sqrt{a^3} \approx 56{,}79$

➔ $C(14{,}77 \mid 56{,}79)$

AUFGABE 3

Weg 1: $y = 2 - x$ mit $0 \leq x \leq 2$ ➔ $y' = -1$

Linienintegral:

$$I_1 = \int_0^2 f(x; y(x)) \cdot \sqrt{1 + (y'(x))^2} dx = \int_0^2 \frac{1}{x^2 + (2-x)^2} \cdot \sqrt{1 + (-1)^2} dx$$

$$I_1 = \int_0^2 \frac{1}{2x^2 - 4x + 4} \cdot \sqrt{2} dx = \frac{\sqrt{2}}{2} \cdot \int_0^2 \frac{1}{x^2 - 2x + 2} dx = \frac{\sqrt{2}}{2} \cdot \int_0^2 \frac{1}{x^2 - 2x + 1 + 1} dx$$

$$I_1 = \frac{\sqrt{2}}{2} \cdot \int_0^2 \frac{1}{(x-1)^2 + 1} dx$$

Substitution: $u = x - 1$ ➔ $x = u + 1$ ➔ $dx = du$

Grenzen: $u_1 = -1$ und $u_2 = 1$

$$I_1 = \frac{\sqrt{2}}{2} \cdot \int_{-1}^1 \frac{1}{u^2 + 1} du = \frac{\sqrt{2}}{2} \cdot [\arctan(u)]_{-1}^1 = \frac{\sqrt{2}}{2} \cdot (\arctan(1) - \arctan(-1))$$

$$I_1 = \frac{\sqrt{2}}{2} \cdot \left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\sqrt{2}}{2} \cdot \frac{\pi}{2} = \frac{\sqrt{2} \cdot \pi}{4} \approx 1{,}11$$

Weg 2: $x = 2 \cdot \sin(t)$; $y = 2 \cdot \cos(t)$ mit $0 \leq t \leq \frac{\pi}{2}$ (Parameterdarstellung)

➔ $\frac{dx}{dt} = x'(t) = 2 \cdot \cos(t)$ und $\frac{dy}{dt} = y'(t) = -2 \cdot \sin(t)$

Linienintegral:

$$I_2 = \int_0^{\frac{\pi}{2}} f(x(t); y(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{1}{4 \cdot (\sin(t))^2 + 4 \cdot (\cos(t))^2} \cdot \sqrt{4 \cdot (\cos(t))^2 + 4 \cdot (\sin(t))^2} dt$$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{1}{4 \cdot ((\sin(t))^2 + (\cos(t))^2)} \cdot \sqrt{4 \cdot ((\sin(t))^2 + (\cos(t))^2)} dt$$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{1}{4} \cdot \sqrt{4} dt = \int_0^{\frac{\pi}{2}} \frac{1}{2} dt = \left[\frac{1}{2} t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} \approx 0{,}79$$

Weg 3: Der Weg 3 besteht aus drei Teilstücken, wobei die Teilstücke 1 und 3 aus Symmetriegründen den gleichen Wert liefern.

Teilstück 1: Strecke von Q zum Punkt $A(0 \mid 1)$

Teilstück 2: Viertelkreis vom Punkt $A(0 \mid 1)$ zum Punkt $B(1 \mid 0)$

Teilstück 3: Strecke von $B(1 \mid 0)$ zum Punkt P

a) Teilstück 3: $y = 0$; $1 \leq x \leq 2$ ➔ $y' = 0$

Linienintegral:

$$I_{3a} = \int_1^2 f(x; y(x)) \cdot \sqrt{1 + (y'(x))^2} dx = \int_1^2 \frac{1}{x^2} \cdot \sqrt{1 + (0)^2} dx = \int_1^2 \frac{1}{x^2} dx$$

$$I_{3a} = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}$$

b) Teilstück 2: $x = \sin(t)$; $y = \cos(t)$ mit $0 \leq t \leq \frac{\pi}{2}$ (Parameterdarstellung)

➔ $\frac{dx}{dt} = x'(t) = \cos(t)$ und $\frac{dy}{dt} = y'(t) = -\sin(t)$

Linienintegral:

$$I_{3b} = \int_0^{\frac{\pi}{2}} f(x(t); y(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$I_{3b} = \int_0^{\frac{\pi}{2}} \frac{1}{(\sin(t))^2 + (\cos(t))^2} \cdot \sqrt{(\cos(t))^2 + (\sin(t))^2} dt$$

$$I_{3b} = \int_0^{\frac{\pi}{2}} \frac{1}{1} \cdot \sqrt{1} dt = \int_0^{\frac{\pi}{2}} 1 dt = [t]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$I_3 = 2 \cdot I_{3a} + I_{3b} = 2 \cdot \frac{1}{2} + \frac{\pi}{2} = 1 + \frac{\pi}{2} \approx 2{,}57$$

Weg 4: Der Weg 4 besteht aus drei Teilstücken, wobei die Teilstücke 1 und 3 aus Symmetriegründen den gleichen Wert liefern.

Teilstück 1: Strecke von Q zum Punkt $C(0 \mid 3)$

Teilstück 2: Viertelkreis vom Punkt $C(0 \mid 3)$ zum Punkt $D(3 \mid 0)$

Teilstück 3: Strecke von $D(3 \mid 0)$ zum Punkt P

a) Teilstück 3: $y = 0$; $2 \leq x \leq 3$ ➔ $y' = 0$

Linienintegral:

$$I_{4a} = \int_2^3 f(x; y(x)) \cdot \sqrt{1 + (y'(x))^2} dx = \int_2^3 \frac{1}{x^2} \cdot \sqrt{1 + (0)^2} dx = \int_2^3 \frac{1}{x^2} dx$$

$$I_{4a} = \left[-\frac{1}{x}\right]_2^3 = -\frac{1}{3} - \left(-\frac{1}{2}\right) = \frac{1}{6}$$

b) Teilstück 2: $x = 3 \cdot \sin(t)$; $y = 3 \cdot \cos(t)$ mit $0 \leq t \leq \frac{\pi}{2}$ (Parameterdarstellung)

➔ $\frac{dx}{dt} = x'(t) = 3 \cdot \cos(t)$ und $\frac{dy}{dt} = y'(t) = -3 \cdot \sin(t)$

Linienintegral:

$$I_{4b} = \int_0^{\frac{\pi}{2}} f(x(t); y(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$I_{4b} = \int_0^{\frac{\pi}{2}} \frac{1}{(9 \cdot \sin(t))^2 + (9 \cdot \cos(t))^2} \cdot \sqrt{(9 \cdot \cos(t))^2 + 9 \cdot (\sin(t))^2}\,dt$$

$$I_{4b} = \int_0^{\frac{\pi}{2}} \frac{1}{9 \cdot ((\sin(t))^2 + (\cos(t))^2)} \cdot \sqrt{9 \cdot ((\sin(t))^2 + (\cos(t))^2)}\,dt$$

$$I_{4b} = \int_0^{\frac{\pi}{2}} \frac{1}{9} \cdot \sqrt{9}\,dt = \int_0^{\frac{\pi}{2}} \frac{1}{3}\,dt = \left[\frac{1}{3}t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{6}$$

$$I_4 = 2 \cdot I_{4a} + I_{4b} = 2 \cdot \frac{1}{6} + \frac{\pi}{6} = \frac{1}{6} \cdot (2 + \pi) \approx 0{,}86$$